MATHEMATICS – MODEL PAPER 3

Time – 02 hours **No. of Pages: 02** **Total Marks- 100**

Answer All

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Q1) (i) A pizza restaurant charges a special price for a large pizza depending on the number of toppings selected. The price $P(n)$, in Euros, is modeled by

$$P(n) = \begin{cases} 12 + 2n, & n < 4, \\ an + b, & n \geq 4, \end{cases}$$

where n is the number of toppings and a, b are constants.
The restaurant changes its pricing policy when the customer selects 4 toppings. To avoid a sudden jump in the price, the manager wants the pricing function to be continuous at $n = 4$.

(a) Determine the relationship between a and b so that the price function is continuous at $n = 4$. **[6 marks]**

(b) Explain what continuity means in this pricing situation. . **[2 marks]**

[Total 25 marks]

(ii) A spherical tumor is growing inside a patient's body. The radius of the tumor after t months is modeled by $r(t) = 0.5e^{0.1t}$ cm, where t is the time in months. The volume of a spherical tumor is given by $V = \frac{4}{3}\pi r^3$.

(a) Using the Chain Rule, find $\frac{dV}{dt}$, the rate of change of the tumor volume with respect to time. **[10 marks]**

(b) Determine the rate at which the tumor volume is increasing after 5 months. **[3 marks]**

(c) Interpret the obtained value in the context of tumor growth. **[2 marks]**

[Total 25 marks]

Q2) A farmer is monitoring the number of cattle in a farm. Let $P(t)$ represent the deviation of the cattle population from its normal sustainable level at time t years. Due to seasonal changes in food supply and environmental conditions, the population satisfies the differential equation

$$P''(t) + 4P'(t) + 5P(t) = 20e^{-t}.$$

Initially, the deviation of the population from its sustainable level is 100 animals, and the initial rate of change is 50 animals per year.

(a) Formulate the initial value problem. **[3 marks]**

(b) Solve the initial value problem using the Laplace transform. **[20 marks]**

(c) Determine the population deviation after 2 years. **[2 marks]**

[Total 25 marks]

Continued…

Q3) An organic farm produces three types of vegetables: tomatoes, beans, and carrots. The production levels in successive seasons are modeled by

$$x_{n+1} = Ax_n,$$

where $x_n = \begin{pmatrix} T_n \\ B_n \\ C_n \end{pmatrix} = \begin{pmatrix} \text{tomato in season n} \\ \text{beans in season n} \\ \text{carrot in season n} \end{pmatrix}$ represents the production levels of tomatoes, beans, and carrots respectively, during season n, and

$$A = \begin{pmatrix} 4 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix}.$$

The matrix A describes how production in one season influences production in the following season.

(a) Determine the characteristic equation of A and hence find its eigenvalues. **[6 marks]**

(b) Find an eigenvector corresponding to each eigenvalue**.** **[12 marks]**

(c) Determine whether A is diagonalizable. **[2 marks]**

(d) If possible, find matrices P and D such that $A = PDP^{-1}$. **[5 marks]**

[Total 25 marks]

Q4) A farmer monitors the height of a rapidly growing vegetable plant. During the early stages of growth, the height $H(t)$, measured in centimetres, is modeled by $H(t) = e^{0.2t}\cos(0.1t)$, where t is the time in weeks.

(a) Obtain Maclaurin series expansion for e^t and $\cos t$. **[4 marks]**

(b) Obtain the Maclaurin series expansion of $e^{0.2t}\cos(0.1t)$ up the term containing t^4. **[10 marks]**

(c) Use the resulting polynomial to estimate the height of the plant at $t = 1$ week. **[3 marks]**

(d) Compare the approximation obtained in Part (c) with the exact value. **[3 marks]**

(e) The farmer wants to estimate the total accumulated growth index of the plant during the first week. Using the Maclaurin polynomial obtained in Part (b), approximate $\int_0^1 H(t)\,dt$. **[5 marks]**

[Total 25 marks]