MATHEMATICS – MODEL PAPER 4

Answer All **Time – 02 hours** **No. of Pages: 02** **Total Marks- 100**

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Q1) (i) Farmer uses an automatic water pump to irrigate a vegetable garden. The water flow rate $Q(t)$, measured in litres per minute, is modeled by $Q(t) = \begin{cases} \frac{t-9}{\sqrt{t+7}-4}, & t \neq 9, \\ k, & t = 9. \end{cases}$ where t is the number of minutes after the pump starts.

At $t = 9$ minutes, the pump undergoes a programmed change in its operating mode. To ensure that the water flow changes smoothly, the engineer wants the flow rate function to be both continuous and differentiable at $t = 9$.

(a) Find the value of k so that $Q(t)$ is continuous at $t = 9$. **[6 marks]**

(b) Determine whether $Q(t)$ is differentiable at $t = 9$. **[10 marks]**

(c) Find the instantaneous rate of change of the water flow at $t = 9$. **[3 marks]**

(ii) A metal rod is heated and then allowed to cool. The temperature difference between the rod and the surrounding environment is modeled by $T(t) = T_0 e^{-kt}$, where $T_0 > 0$ is the initial temperature difference and $k > 0$ is a cooling constant. The average rate of temperature change during the time interval $[0, t]$ is given by

$$A(t) = \frac{T(t) - T(0)}{t}.$$

Determine the instantaneous rate of temperature change at the moment $t = 0$ by evaluating $\lim_{t \to 0} A(t)$. **[6 marks]**

[Total 25 marks]

Q2) A glass-bottom sightseeing boat travels toward a coral reef. The vertical displacement $y(t)$of the boat caused by the combined effect of wave motion and a periodic underwater current is modeled by

$\frac{d^2y}{dt^2} + 4y = 6\cos(2t) + 3t\sin(2t)$, where $y(t)$ is the vertical displacement in meters and t is the time in minutes.

(a) Find the complementary solution. **[5 marks]**

(b) Show that the two solutions $y_1 = \cos 2t$, $y_2 = \sin 2t$ are independent. **[3 marks]**

(c) Using the method of variation of parameters, determine the functions $u_1(t)$ and $u_2(t)$. **[12 marks]**

(d) Hence, find the particular solution y_p and simplify your answer. **[5 marks]**

[Total 25 marks]

Continued…

Q3) During a beach activity, a group of children builds several sand houses along a straight section of the beach. The width of the sandy area occupied by their constructions changes with the distance x from the starting point according to $w(x) = 0.5 + 0.2x - \frac{0.02}{3}x^3$, where $w(x)$ is measured in meters and x is measured in meters. Assume that the children occupy the beach from $x = 0$ to $x = 2$ meters.

(a) Determine the total area of the beach covered by the children's sand house. **[12 marks]**

(b) Determine the rate at which the width of the covered region changes at $x = 1$. **[2 marks]**

(c) Find the position x at which the width of the covered area is maximum. **[5 marks]**

(d) Determine the maximum width of the covered region. **[2 marks]**

(e) Find the average width of the covered region over the 2-metre section. **[4 marks]**

[Total 25 marks]

Q4) To keep a computer processor from overheating, engineers attach metal cooling fins to conduct heat away into the air. Let x be the distance (in centimeters) along a metal fin of length $L = \ln(2)$ cm. Under steady-state conditions, the temperature $T(x)$ of the fin (measured in degrees Celsius above room temperature) is modeled by the second-order linear Boundary Value Problem (BVP):

$$T''(x) - 4T(x) = 0$$

At the hot processor base ($x = 0$), the temperature is fixed. At the tip of the fin ($x = \ln(2)$), the temperature is cooler because heat has escaped into the surrounding air.

(a) Write down the characteristic (auxiliary) equation for the given differential equation. **[5 marks]**

(b) Find the general solution $T(x)$ in terms of constants C_1 and C_2. **[3 marks]**

(c) Suppose we measure the temperature at both ends of the fin. Let the temperature at the base ($x = 0$): $T(0) = 4.25°\text{C}$ and at the tip ($x = \ln(2)$): $T(\ln 2) = 2°\text{C}$. Set up a system of two equations to solve the constants C_1 and C_2. **[8 marks]**

(d) Find the unique temperature profile along the fin. **[5 marks]**

(e) Find the temperature at the mid-point of the fin. **[4 marks]**

[Total 25 marks]