MATHEMATICS – MODEL PAPER 6-

Answer All **Time – 02 hours** **No. of Pages: 06** **Total Marks- 100**

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Q1) (i) A data science team is analyzing the performance of a machine using three sensor measurements: temperature**,** vibration, and energy consumption. The measurements are represented by vectors in $\mathbb{R}^3$. For two consecutive observations, the data vectors are $\underline{u} = (2,3,4)$ and $\underline{v} = (1,2,3)$.

(a) Express the vector $\underline{u}$ as a linear combination of the vectors: $\underline{w_1} = (1,0,1)$, $\underline{w_2} = (0,1,1)$, $\underline{w_3} = (1,1,0)$. **[5 marks]**

(b) Find the Euclidean norm (magnitude) of the vector $\underline{u}$. **[2 marks]**

(c) Normalize $\underline{u}$ to obtain a unit vector in the same direction. **[2 marks]**

(d) Calculate the vector product (cross product) $\underline{u} \times \underline{v}$. **[3 marks]**

(e) Verify that $\underline{u} \times \underline{v}$ is perpendicular to both $\underline{u}$ and $\underline{v}$. **[4 marks]**

(ii) An embassy processes three categories of non-student visas: Residential visa, Tourist visa and Dependent visa. Let x, y, z be the number of Residential, Tourists, and Dependent visa applicants respectively. During a particular month, the embassy records the following information:

The following relation exists between the three visa categories:

$$x + 2y - z = 3$$

The second administrative report indicate the below relationship:

$$2x + 5y + z = 7$$

A third report, initiate a policy parameter k, and the below relation exist among three visa categories:
$x + y - k^2 z = -2$, where k represents a policy adjustment factor related to dependent visa processing.

(a) Represent the above information in augmented matrix. **[1 marks]**

(b) Use Gaussian elimination to reduce the augmented matrix to row echelon form. **[6 marks]**

(c) Determine the values of k for which the embassy's visa applicant data has a unique solution. **[2 marks]**

[Total 25 marks]

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Q2) A data scientist models the processing time $T(x)$(in milliseconds) required by an algorithm as $T(x) = \frac{x-5}{\sqrt{3x+10}-5}$, where x represents the number of data batches processed, and x is close to 5.

(a) As the number of data batches approaches 5, determine the limiting processing-time factor $\lim_{x\to 5} \frac{x-5}{\sqrt{3x+10}-5}$ without using L'Hôpital's Rule. **[5 marks]**

(b) Using your answer to Part (a), define a new function

$$f(x) = \begin{cases} \frac{x-5}{\sqrt{3x+10}-5}, & x \neq 5, \\ k, & x = 5. \end{cases}$$

Find the value of k that makes $f(x)$ continuous at $x = 5$. **[5 marks]**

(c) Determine the equation of the tangent line to the curve $y = \sqrt{3x + 10}$ at the point where $x = 5$. **[5 marks]**

(d) To study how small changes in the input data size affect the algorithm's prediction error, the data scientist defines a sensitivity function $R(x) = \frac{x\ln(x)-x+1}{(x-1)^2}$, where x represents a small change in the normalized input size. Evaluate $\lim_{x\to 1} R(x)$ using L'Hôpital's Rule. **[5 marks]**

(e) The total computational cost of the algorithm is modeled by $C(x) = x^2 - 12x + 50$, where x is the number of data batches. Determine the number of batches that minimizes the computational cost. **[5 marks]**

[Total 25 marks]

Q3) A remote island with a bird sanctuary experiences an outbreak of bird flu. Health authorities monitor the epidemic and implement a vaccination and quarantine program to reduce the spread of the disease.
Let $I(t)$ be the number of infected birds at time t(days), and $S(t)$be the number of susceptible birds at time t.
Following observations were identified during field study:

- infected birds spread the disease at a rate proportional to the current number of infected birds with proportionality constant $0.18\ \text{day}^{-1}$;
- vaccination and quarantine remove infected birds at a constant rate of 45 birds per day;
- susceptible birds are vaccinated continuously at a rate proportional to the remaining susceptible population with proportionality constant $0.05\ \text{day}^{-1}$.
- Initially, there are 300 infected birds and 2400 susceptible birds.

(a) Formulate the initial value problems that model the infected and susceptible bird populations. **[5 marks]**

(b) Solve the initial value problem that models the number of infected birds on the island. Hence, obtain the number of infected birds as a function of time t.

[10 marks]

(c) Determine the long-term behavior of the infected bird population. **[2 marks]**

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(d) Solve the initial value problem for the susceptible bird population and obtain an expression for $S(t)$. **[4 marks]**

(e) Compare the behavior of the infected population $I(t)$ and susceptible population $S(t)$ over time. Explain how the vaccination and quarantine measures influence the spread of bird flu on the island. **[4 marks]**

[Total 25 marks]

Q4) **(i)** During the autumn season in southern Germany, the rate at which apples are harvested from a green house is modeled by

$$A'(t) = 500 + \sqrt{t} \quad kg \text{ per hour},$$

where t is the time (in hour) measured from the beginning of the day's harvest ($0 \leq t \leq 8$), and $A(t)$ denotes the total amount of apples harvested (in kilogram) by time t.

(a) Using the Fundamental Theorem of Calculus, express the total quantity of apples harvested during the first four hours as a definite integral. **[5 marks]**

(b) Evaluate the integral to determine the total quantity of apples harvested during the first four hours. **[5 marks]**

(ii) The increase in the height of a plant produced by applying a dose D(in grams) of fertilizer is modeled by

$$H(D) = \left(\frac{C}{3} - \frac{D}{4}\right) D^2,$$

where $H(D)$is the increase in height (in centimeters) and $C$is a constant representing the soil condition and plant type.

(a) Find the rate of change of plant height with respect to the fertilizer dose, $\frac{dH}{dD}$. **[5 marks]**

(b) Determine the fertilizer dose that produces the maximum increase in plant height. **[6 marks]**

(c) Find the maximum possible increase in height. **[4 marks]**

[Total 25 marks]