MATHEMATICS – MODEL PAPER 7

Answer All **Time – 02 hours** **No. of Pages: 3** **Total Marks- 100**

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Q1) (i) A sphere is dropped from a height of 10 m and rebounds to a fraction r of its previous height. The total distance travelled before coming to rest is 82 m.

(a) Form an equation in r. **[4 marks]**

(b) Determine the rebound ratio. **[3 marks]**

(c) Hence determine the height reached after the sixth bounce. **[3 marks]**

(d) Estimate the total distance travelled after the first six impacts and the error in approximating the infinite distance by this finite value. **[3 marks]**

(ii) A biology student studying the growth resistance of a newly introduced plant species in a controlled environment. The rate of change of a measured environmental response is modeled by $R(x) = \frac{1}{1+x^5}$, where x represents a scaled distance from the center of the observation region. The total response accumulated from the center of the region to a distance of 0.5 units is given by

$$A = \int_0^{0.5} \frac{1}{1+x^5}\, dx.$$

(a) Express $R(x)$ as a power series and hence obtain a power series representation for the accumulated response

$$\int \frac{1}{1+x^5}\, dx.$$

State the interval of convergence of the series. **[5 marks]**

(b) Use the power series obtained in Part (a) to approximate the accumulated response $A = \int_0^{0.5} \frac{1}{1+x^5}\, dx$ with an error less than 10^{-6}. Clearly justify the number of terms used.

[7 marks]
[Total 25 marks]

(Q2) (i) In a mathematical model of a temperature profile along a one-dimensional sensor, the temperature variations are represented by linear polynomials. The set of all linear temperature profiles on the interval [0,1] is considered as a vector space V with all polynomials of degree at most 1. The similarity between two temperature profiles $f(t)$ and $g(t)$is measured using the inner product $< f, g >= \int_0^1 f(t)g(t)\, dt$, where t represents the normalized position along the sensor. Two temperature profiles are given by $f(t) = t + 3$ and $g(t) = 3t - 1$.

(a) Calculate the inner product $< f, g >$, which represents the correlation between the two temperature profiles. **[4 marks]**

(b) Determine the norm $\| f \|$ of the temperature profile $f(t)$, which represents its magnitude. **[4 marks]**

(c) Find the distance $d(f, g)$ between the two temperature profiles. Interpret what this distance represents in the context of temperature profiles. **[5 marks]**

(ii) In a structural engineering model, the energy distribution of a system is represented by a symmetric matrix A. The matrix is given by $\begin{pmatrix} 1 & 2 & -3 \\ 2 & 5 & -4 \\ -3 & -4 & 8 \end{pmatrix}$.

Find the nonsingular matrix P such that $P^T AP$ is diagonal and find $P^T AP$.

[12 marks]

[Total 25 marks]

Q3) **(i)** A researcher studies the growth pattern of a young sugar cane plant in a closed agricultural field. The height contribution of the plant along a normalized section of the stem is modeled by the function $h(x) = x\sqrt{1-x}$, where x represents the position along the stem, with $0 \leq x \leq 1$. The total growth index of the sugar cane stem section is estimated by $A = \int_0^1 x\sqrt{1-x}\, dx$.

(a) Using suitable substitution, transform the integral into an equivalent form. **[4 marks]**

(b) Evaluate the integral obtained in Part (a), using integration by parts. **[6 marks]**

(ii) A fisherman is catching fish during the first hour after sunrise. Let x denote the time (in hours) after sunrise, where $0 \leq x \leq 1$. Suppose that the effective fishing rate (in kg/h) is modeled by $R(x) = e^x \cos^2(\pi x)$. The factor e^x represents the increasing fish activity during the first hour, while $\cos^2(\pi x)$ represents the effect of sea waves on the efficiency of the fishing net.

(a) Explain the meaning of the function $R(x)$ in the given context. **[1 marks]**

(b) Write an integral that represents the total amount of fish caught during the first hour. **[2 marks]**

(c) Evaluate the total amount of fish caught during the first hour. **[12 marks]**

[Total 25 marks]

Q4) **(i)** A coastal ecologist is studying the growth of two dominant plant communities in a mangrove ecosystem located near a sheltered bay. The first community consists of red mangroves, while the second consists of salt-tolerant coastal shrubs growing in the surrounding marshland.

(a) Apply the Laplace transform to the system and obtain the algebraic equations for $Y_1(s)$and $Y_2(s)$. **[5 marks]**

(b) Solve the transformed system obtained in Part(a) to find $Y_1(s)$and $Y_2(s)$. **[8 marks]**

(c) Using partial fractions and inverse Laplace transforms, determine $y_1(t)$and $y_2(t)$.

[12 marks]
[Total 25 marks]